

ELLIPTICITY AND ERGODICITY

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ABSTRACT. Let $S = \{S_t\}_{t \geq 0}$ be the submarkovian semigroup on $L_2(\mathbf{R}^d)$ generated by a self-adjoint, second-order, divergence-form, elliptic operator H with Lipschitz continuous coefficients c_{ij} . Further let Ω be an open subset of \mathbf{R}^d . Under the assumption that $C_c^\infty(\mathbf{R}^d)$ is a core for H we prove that S leaves $L_2(\Omega)$ invariant if, and only if, it is invariant under the flows generated by the vector fields $Y_i = \sum_{j=1}^d c_{ij} \partial_j$.

1. INTRODUCTION

Let S be a submarkovian semigroup on $L_2(\mathbf{R}^d)$ generated by a self-adjoint second-order elliptic operator H in divergence form. If the operator is strongly elliptic then S acts ergodically, i.e. there are no non-trivial S -invariant subspaces of $L_2(\mathbf{R}^d)$. Nevertheless there are many examples of degenerate elliptic operators for which there are subspaces $L_2(\Omega)$ invariant under the action of S (see, for example, [ERSZ06] [RS07] [ER07]). Our aim is to examine operators with coefficients which are Lipschitz continuous and characterize the S -invariance of $L_2(\Omega)$ by the invariance under a family of associated flows. Then one can combine the characterization with a domination estimate to establish invariance properties for a large class of degenerate operators with L_∞ -coefficients. In order to formulate our main result more precisely we need some further notation.

First define H as the Friedrichs extension of the positive symmetric operator H_0 with domain $D(H_0) = C_c^\infty(\mathbf{R}^d)$ and action

$$H_0 \varphi = - \sum_{i,j=1}^d \partial_i c_{ij} \partial_j \varphi$$

where the coefficients $c_{ij} = c_{ji} \in W_{\text{loc}}^{1,\infty}(\mathbf{R}^d)$ are real and $C = (c_{ij})$ is a positive-definite matrix over \mathbf{R}^d . Thus H is a divergence form operator with coefficients which are locally Lipschitz continuous. It is the positive, self-adjoint, operator on $L_2(\mathbf{R}^d)$ associated with the closure \bar{h}_0 of the quadratic form

$$(1) \quad h_0(\varphi) = \sum_{i,j=1}^d (\partial_i \varphi, c_{ij} \partial_j \varphi)$$

with domain $D(h_0) = C_c^\infty(\mathbf{R}^d)$. Set $h = \bar{h}_0$. Then h is a Dirichlet form and the self-adjoint semigroup S generated by H is automatically submarkovian (for details on Dirichlet forms and submarkovian semigroups see [FOT94] or [BH91]).

Secondly, let $\psi \in C_c^\infty(\mathbf{R}^d)$ and define Y_ψ as the L_2 -closure of the first-order partial differential operator with action

$$Y_\psi \varphi = \sum_{i,j=1}^d c_{ij} (\partial_i \psi) \partial_j \varphi$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. The Y_ψ generate positive, continuous, one-parameter quasi-contractive groups T^ψ on $L_2(\mathbf{R}^d)$. The latter result follows because the coefficients $c_j = \sum_{i=1}^d c_{ij}(\partial_i \psi)$ of the Y_ψ are in $W^{1,\infty}(\mathbf{R}^d)$ (see Lemma 3 for details). We refer to these groups as flows.

Our primary aim is to establish the following characterization of invariance.

Theorem 1. *Assume $c_{ij} \in W_{\text{loc}}^{1,\infty}(\mathbf{R}^d)$ and that $C_c^\infty(\mathbf{R}^d)$ is a core for H . Let Ω be a measurable subset of \mathbf{R}^d .*

The following conditions are equivalent.

- I. $S_t L_2(\Omega) \subseteq L_2(\Omega)$ for all $t > 0$.
- II. $T_t^\psi L_2(\Omega) = L_2(\Omega)$ for all $\psi \in C_c^\infty(\mathbf{R}^d)$ and all $t \in \mathbf{R}$.

If the coefficients $c_{ij} \in W^{1,\infty}(\mathbf{R}^d)$ one can characterize the S -invariance of $L_2(\Omega)$ with the flows generated by the L_2 -closures of the operators Y_i with action

$$Y_i \varphi = \sum_{j=1}^d c_{ij} \partial_j \varphi$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ where $i \in \{1, \dots, d\}$.

Theorem 2. *Assume $c_{ij} \in W^{1,\infty}(\mathbf{R}^d)$ and that $C_c^\infty(\mathbf{R}^d)$ is a core for H . Let Ω be a measurable subset of \mathbf{R}^d .*

The following conditions are equivalent:

- I. $S_t L_2(\Omega) \subseteq L_2(\Omega)$ for all $t > 0$,
- II. $T_t^{(i)} L_2(\Omega) = L_2(\Omega)$ for all $i \in \{1, \dots, d\}$ and all $t \in \mathbf{R}$.

The operators Y_i were used by Oleinik and Radkevich [OR73] to analyze hypoellipticity and subellipticity properties of degenerate elliptic operators H with C^∞ -coefficients c_{ij} . In particular they established that if the Y_i satisfy Hörmander's condition [Hör67], i.e. if the Y_i and their multi-commutators up to some finite order k span the tangent space at each point $x \in \mathbf{R}^d$, then H satisfies the subelliptic estimate $H \geq \mu \Delta^\gamma - \nu I$ for some $\mu > 0$, $\nu \geq 0$ and $\gamma \in (0, 1]$ where Δ is the usual Laplacian. There is, however, no simple relationship between the values of k and γ (see [JSC86] for a review of related results).

The foregoing characterizations of S -invariant subspaces differ from the earlier results [RS07] [ER07] based on capacity estimates on the boundary $\partial\Omega$ of Ω . Here our results require the assumption that $C_c^\infty(\mathbf{R}^d)$ is a core for H . Unfortunately, this assumption does not always hold for degenerate elliptic operators, see [RS09]. We do not know if this supposition is indeed necessary.

2. PRELIMINARIES

The proofs of Theorems 1 and 2 depend on some basic properties of flows which follow from standard results on first-order partial differential equations which we first summarize.

Let $c_j \in W^{1,\infty}(\mathbf{R}^d)$ for each $j \in \{1, \dots, d\}$ and let $Y = \sum_{j=1}^d c_j \partial_j$ be the corresponding first-order operator on $C_1(\mathbf{R}^d)$. The operator Y is a model for the operators Y_ψ and Y_i introduced in Section 1. Since $c_j \in W^{1,\infty}(\mathbf{R}^d)$ there exists a unique Lipschitz continuous function $(t, x) \in \mathbf{R} \times \mathbf{R}^d \mapsto f_t(x) \in \mathbf{R}^d$ satisfying the evolution equation

$$(2) \quad \frac{\partial f_t(x)}{\partial t} = c(f_t(x)) \quad \text{and} \quad f_0(x) = x$$

for all $x \in \mathbf{R}^d$ and $t \in \mathbf{R}$ (see, for example, [Hil69] Chapter 2 and in particular Theorem 2.6.3). Moreover, there are $M, \omega \geq 0$ such that

$$|f_t(x) - f_s(x)| \leq M |t - s| e^{\omega(|t| \wedge |s|)}$$

and

$$|f_t(x) - f_t(y)| \leq |x - y| e^{\omega|t|}$$

for all $x, y \in \mathbf{R}^d$ and $s, t \in \mathbf{R}$ (see [Hil69], Theorems 3.1.1 and 3.2.1). The partial derivatives $\partial_i(f_t)_j$ of the components $(f_t)_j$ of f_t and the Jacobian J_t of the transformation $x \in \mathbf{R}^d \rightarrow f_t(x)$ are bounded. Moreover, there are $M' \geq 1$ and $\omega' \geq 0$ such that

$$(3) \quad \|J_t\|_\infty = \|\det(\partial_i(f_t)_j)\|_\infty \leq M' e^{\omega'|t|}$$

for all $t \in \mathbf{R}$. We adopt the conventional notation $\exp(tY)(x) = f_t(x)$ and then one has the group property

$$(\exp(tY)(\exp(sY)))(x) = \exp((t+s)Y)(x)$$

for all $x \in \mathbf{R}^d$ and $t, s \in \mathbf{R}$.

One can immediately define a positive, continuous, one-parameter group of isometries T_t on $C_b(\mathbf{R}^d)$ by setting $(T_t\psi)(x) = \psi(\exp(tY)(x)) = (\psi \circ f_t)(x)$. In particular T is conservative, i.e. $T_t\mathbb{1} = \mathbb{1}$ for all $t \in \mathbf{R}$. Moreover, the group T extends to a conservative weakly* continuous group of isometries on $L_\infty(\mathbf{R}^d)$. But the Jacobian of the transformation $x \rightarrow e^{tY}x$ is uniformly bounded by (3). Therefore T also extends to a strongly continuous one-parameter group on the spaces $L_p(\mathbf{R}^d)$ for $p \in [1, \infty)$. In fact one has the following statement in which the weak* topology is to be understood if $p = \infty$.

Lemma 3. *The L_p -closure of $Y|_{C_c^\infty(\mathbf{R}^d)}$ generates a positive, continuous, one-parameter group $T^{(p)}$ on $L_p(\mathbf{R}^d)$, for each $p \in [1, \infty]$, with action*

$$(4) \quad (T_t^{(p)}\psi)(x) = \psi(\exp(tY)(x))$$

for $\psi \in L_p(\mathbf{R}^d)$ and $t \in \mathbf{R}$. Moreover, $T_t^{(p)}W^{1,p}(\mathbf{R}^d) = W^{1,p}(\mathbf{R}^d)$ and

$$(5) \quad \|T_t^{(p)}\|_{p \rightarrow p} \leq e^{\nu|t|/p}$$

for all $t \in \mathbf{R}$ with $\nu = \|\operatorname{div} c\|_\infty$.

Proof. First it follows from the definition (4) that $T^{(p)}$ is a positive, strongly continuous, group on $L_p(\mathbf{R}^d)$ if $p \in [1, \infty)$ and $T^{(\infty)}$ is a positive, weakly* continuous, group of isometries on $L_\infty(\mathbf{R}^d)$.

Secondly, $W^{1,p}(\mathbf{R}^d)$ is $T^{(p)}$ -invariant because the bounds on the partial derivatives $\partial_i(f_t)_j$ immediately give bounds

$$\|\partial_i T_t^{(p)}\psi\|_p \leq M e^{\omega|t|} \sup_{1 \leq j \leq d} \|\partial_j \psi\|_p$$

for all $\psi \in W^{1,p}(\mathbf{R}^d)$.

Thirdly, it follows from the definition (4) that the generator of $T^{(p)}$ is a closed extension of the operator Y defined on $W^{1,p}(\mathbf{R}^d)$. But since $W^{1,p}(\mathbf{R}^d)$ is $T^{(p)}$ -invariant it must be a core of the generator. Hence the generator Y_p is the L_p -closure of $Y|_{W^{1,p}(\mathbf{R}^d)}$. Then Y_p is the L_p -closure of $Y|_{C_c^\infty(\mathbf{R}^d)}$ since $C_c^\infty(\mathbf{R}^d)$ is dense in $W^{1,p}(\mathbf{R}^d)$.

Fourthly, the adjoint Y_∞^* of Y_∞ generates a strongly continuous group of isometries on $L_1(\mathbf{R}^d)$. But $Y_\infty^*\psi = -Y\psi - (\operatorname{div} c)\psi$ for all $\psi \in W^{1,1}(\mathbf{R}^d)$. Therefore $Y_\infty^* \supseteq -Y_1 - (\operatorname{div} c)I$ and since both operators are generators of continuous groups one must have $Y_\infty^* = -Y_1 - (\operatorname{div} c)I$. Then the group $T^{(1)}$ generated by $Y_1 = -Y_\infty^* - (\operatorname{div} c)I$ satisfies the bounds $\|T_t^{(1)}\|_{1 \rightarrow 1} \leq \exp(\nu|t|)$ for all $t \in \mathbf{R}$ by the Trotter product formula. The L_p -bounds (5) follow by interpolation. \square

The functions $L_\infty(\mathbf{R}^d)$ act as multipliers on the L_p -spaces and it follows from the action (4) of the group that

$$(6) \quad T_t^{(p)}(\varphi\psi) = (T_t^{(\infty)}\varphi)(T_t^{(p)}\psi)$$

for all $\varphi \in L_\infty(\mathbf{R}^d)$ and $\psi \in L_p(\mathbf{R}^d)$.

Lemma 4. *Let Ω be a measurable subset of \mathbf{R}^d . The following conditions are equivalent.*

- I. $T_t^{(p)}L_p(\Omega) = L_p(\Omega)$ for all $t \in \mathbf{R}$ and for one (for all) $p \in [1, \infty)$.
- II. $[T_t^{(p)}, \mathbb{1}_\Omega] = 0$ for all $t \in \mathbf{R}$ and for one (for all) $p \in [1, \infty)$.
- III. $T_t^{(\infty)}\mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $t \in \mathbf{R}$.

Proof. First suppose Condition I is valid for one $p \in [1, \infty)$. Then if $q \in [1, \infty)$ one has

$$T_t^{(q)}(L_p(\Omega) \cap L_q(\Omega)) = T_t^{(p)}(L_p(\Omega) \cap L_q(\Omega)) \subseteq L_p(\Omega) \cap L_q(\mathbf{R}^d) = L_p(\Omega) \cap L_q(\Omega)$$

for all $t \in \mathbf{R}$. Then since $L_p(\Omega) \cap L_q(\Omega)$ is dense in $L_q(\Omega)$ one has $T_t^{(q)}L_q(\Omega) \subseteq L_q(\Omega)$ for all $t \in \mathbf{R}$ and by the group property $T_t^{(q)}L_q(\Omega) = L_q(\Omega)$ for all $t \in \mathbf{R}$. Thus Condition I is valid for all $p \in [1, \infty)$.

Similarly, since $[T_t^{(p)}, \mathbb{1}_\Omega]\psi = [T_t^{(q)}, \mathbb{1}_\Omega]\psi$ for all $\psi \in L_p(\mathbf{R}^d) \cap L_q(\mathbf{R}^d)$ it follows that if Condition II is valid for one $p \in [1, \infty)$ then it is valid for all $p \in [1, \infty)$.

Now it suffices to prove the equivalence of the three conditions with $p = 2$.

I \Rightarrow II Since $\mathbb{1}_\Omega$ is the orthogonal projection from $L_2(\mathbf{R}^d)$ onto the invariant subspace $L_2(\Omega)$ it follows that

$$T_t^{(2)}\mathbb{1}_\Omega = \mathbb{1}_\Omega T_t^{(2)}\mathbb{1}_\Omega.$$

Then by taking adjoints

$$\mathbb{1}_\Omega T_t^{(2)*} = \mathbb{1}_\Omega T_t^{(2)*}\mathbb{1}_\Omega.$$

But since the generator Y_2 of $T^{(2)}$ satisfies $Y_2 = -Y_2^* - (\operatorname{div} c)I$ it follows by another application of the Trotter product formula that

$$\mathbb{1}_\Omega T_t^{(2)} = \mathbb{1}_\Omega T_t^{(2)}\mathbb{1}_\Omega.$$

Therefore $[T_t^{(2)}, \mathbb{1}_\Omega] = 0$.

II \Rightarrow III It follows immediately from Condition II and (6) that

$$\mathbb{1}_\Omega T_t^{(2)}\psi = T_t^{(2)}\mathbb{1}_\Omega\psi = (T_t^{(\infty)}\mathbb{1}_\Omega)T_t^{(2)}\psi$$

for all $\psi \in L_2(\mathbf{R}^d)$. Replacing ψ by $T_{-t}^{(2)}\psi$ one deduces that $T_t^{(\infty)}\mathbb{1}_\Omega = \mathbb{1}_\Omega$.

III \Rightarrow I If $\psi \in L_2(\mathbf{R}^d)$ then

$$T_t^{(2)}\mathbb{1}_\Omega\psi = (T_t^{(\infty)}\mathbb{1}_\Omega)T_t^{(2)}\psi = \mathbb{1}_\Omega T_t^{(2)}\psi$$

by (6) and Condition III. Therefore $T_t^{(2)}L_2(\Omega) \subseteq L_2(\Omega)$ for all $t \in \mathbf{R}$. Since $T^{(2)}$ is a group it then follows that $T_t^{(2)}L_2(\Omega) = L_2(\Omega)$ for all $t \in \mathbf{R}$. \square

Lemma 4 has the following straightforward corollary.

Corollary 5. *The following conditions are equivalent.*

- I. $T_t^{(2)}L_2(\Omega) = L_2(\Omega)$ for all $t \in \mathbf{R}$.
- II. $(Y^*\varphi, \mathbb{1}_\Omega) = 0$ for all $\varphi \in C_c^\infty(\mathbf{R}^d)$.

Proof. I \Rightarrow II It follows from Lemma 4 that Condition I is equivalent to $T_t^{(\infty)} \mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t \in \mathbf{R}$ and this is clearly equivalent to

$$(T_t^{(\infty)*} \varphi, \mathbf{1}_\Omega) = (\varphi, \mathbf{1}_\Omega)$$

for all $t \in \mathbf{R}$ and $\varphi \in L_1(\mathbf{R}^d)$. Then Condition II follows by differentiation.

II \Rightarrow I Since $Y^* = -Y - (\operatorname{div} c)I$ and $C_c^\infty(\mathbf{R}^d)$ is a core of Y_1 it follows from Condition II by closure that $((Y_1 + (\operatorname{div} c)I)\varphi, \mathbf{1}_\Omega) = 0$ for all $\varphi \in D(Y_1)$. But $(Y_1 + (\operatorname{div} c)I) = -Y_\infty^*$. Therefore Condition II implies $(Y_\infty^* \varphi, \mathbf{1}_\Omega) = 0$ for all $\varphi \in D(Y_\infty^*) = D(Y_1)$. Now Duhamel's formula gives

$$(T_t^{(\infty)*} \varphi, \mathbf{1}_\Omega) - (\varphi, \mathbf{1}_\Omega) = \int_0^t ds (Y_\infty^* T_s^{(\infty)*} \varphi, \mathbf{1}_\Omega) = 0$$

for all $\varphi \in D(Y_\infty^*)$. Then since $D(Y_\infty^*) = D(Y_1)$ is dense in $L_1(\mathbf{R}^d)$ it follows that $T_t^{(\infty)} \mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t \in \mathbf{R}$. Condition I follows from Lemma 4. \square

3. PROOFS OF THEOREMS 1 AND 2

First recall that h denotes the Dirichlet form associated with the elliptic operator H , i.e. h is the closure of the form h_0 defined by (1). The form H is local in the sense of [FOT94], i.e. if $\varphi, \psi \in D(h)$ and $\varphi\psi = 0$ then the sesquilinear form associated with h satisfies $h(\varphi, \psi) = 0$. (This condition appears somewhat stronger than that of [FOT94] but it is in fact equivalent by a result of Schmuland [Sch95].) Secondly, it follows from the Dirichlet property that $D(h) \cap L_\infty(\mathbf{R}^d)$ is an algebra. Therefore, for each positive $\xi \in D(h) \cap L_\infty(\mathbf{R}^d)$, one can define the truncation h_ξ of h by

$$(7) \quad h_\xi(\varphi, \psi) = 2^{-1} (h(\xi\varphi, \psi) + h(\varphi, \xi\psi) - h(\xi, \varphi\psi))$$

for all $\varphi, \psi \in D(h) \cap L_\infty(\mathbf{R}^d)$. The Dirichlet property implies that $D(h) \subseteq D(h_\xi)$ and

$$0 \leq h_\xi(\varphi) \leq \|\xi\|_\infty h(\varphi)$$

for all $\varphi \in D(h)$ (see, for example, [BH91], Proposition 4.1.1). The truncated form h_ξ is not necessarily closed but it is local. The locality of h_ξ follows straightforwardly from the locality of h .

Thirdly, for each $\psi \in C_c^\infty(\mathbf{R}^d)$ let Y_ψ denote the vector field on \mathbf{R}^d defined in Section 1. The operators Y_ψ are related to the truncated forms h_ξ by the identity

$$(8) \quad (\xi, Y_\psi \varphi) = \int_{\mathbf{R}^d} dx \xi(x) \sum_{i,j=1}^d c_{ij}(x) (\partial_j \psi)(x) (\partial_i \varphi)(x) = h_\xi(\psi, \varphi)$$

which is certainly valid for all $\varphi, \psi \in C_c^\infty(\mathbf{R}^d)$ and positive $\xi \in D(h) \cap L_\infty(\mathbf{R}^d)$. But

$$|(\xi, Y_\psi \varphi)| \leq \|\xi\|_\infty \|Y_\psi \varphi\|_1$$

$$\begin{aligned} &= \|\xi\|_\infty \int_{\mathbf{R}^d} dx \left| \sum_{i,j=1}^d c_{ij}(x) (\partial_i \psi)(x) (\partial_j \varphi)(x) \right| \\ &\leq \|\xi\|_\infty \int_{\mathbf{R}^d} dx \left(\sum_{i,j=1}^d c_{ij}(x) (\partial_i \psi)(x) (\partial_j \psi)(x) \right)^{1/2} \left(\sum_{i,j=1}^d c_{ij}(x) (\partial_i \varphi)(x) (\partial_j \varphi)(x) \right)^{1/2} \\ &\leq \|\xi\|_\infty h(\psi)^{1/2} h(\varphi)^{1/2}. \end{aligned}$$

Therefore (8) extends by continuity to all $\varphi, \psi \in D(h) \cap L_\infty(\mathbf{R}^d)$ always with $\xi \in D(h) \cap L_\infty(\mathbf{R}^d)$ positive. Moreover, it follows from this estimation that

$$\|Y_\psi \varphi\|_1 \leq h(\psi)^{1/2} h(\varphi)^{1/2}$$

for all $\varphi, \psi \in C_c^\infty(\mathbf{R}^d)$ and then by continuity for all $\varphi \in D(h)$. In particular $Y_\psi D(h) \subseteq L_1(\mathbf{R}^d)$ for all $\psi \in C_c^\infty(\mathbf{R}^d)$.

Now we are prepared to prove the first theorem.

Proof of Theorem 1 I \Rightarrow II Note that the proof of this implication does not use the assumption that $C_c^\infty(\mathbf{R}^d)$ is a core of H .

It follows from Condition I that $\mathbb{1}_\Omega D(h) \subseteq D(h)$ (see [FOT94], Theorem 1.6.1). Thus if $\xi \in D(h) \cap L_\infty(\mathbf{R}^d)$ is positive then $\mathbb{1}_\Omega \xi \in D(h) \cap L_\infty(\mathbf{R}^d)$ and $\mathbb{1}_\Omega \xi$ is also positive. Now it follows from (8) that

$$(\xi, \mathbb{1}_\Omega Y_\psi \varphi) = (\mathbb{1}_\Omega \xi, Y_\psi \varphi) = h_{\mathbb{1}_\Omega \xi}(\psi, \varphi)$$

for all $\psi \in C_c^\infty(\mathbf{R}^d)$ and $\varphi \in D(h) \cap L_\infty(\mathbf{R}^d)$. Then by (7) and locality

$$\begin{aligned} h_{\mathbb{1}_\Omega \xi}(\psi, \varphi) &= 2^{-1} (h((\mathbb{1}_\Omega \xi)\psi, \varphi) + h(\psi, (\mathbb{1}_\Omega \xi)\varphi)) - h(\mathbb{1}_\Omega \xi, \psi\varphi) \\ &= 2^{-1} \left(h(\mathbb{1}_\Omega(\xi\psi), \mathbb{1}_\Omega \varphi) + h(\mathbb{1}_\Omega \psi, \mathbb{1}_\Omega(\xi\varphi)) - h(\mathbb{1}_\Omega \xi, \mathbb{1}_\Omega \psi\varphi) \right) \\ &= 2^{-1} \left(h(\xi(\mathbb{1}_\Omega \psi), \mathbb{1}_\Omega \varphi) + h(\mathbb{1}_\Omega \psi, \xi(\mathbb{1}_\Omega \varphi)) - h(\xi, (\mathbb{1}_\Omega \psi)(\mathbb{1}_\Omega \varphi)) \right) \\ &= h_\xi(\mathbb{1}_\Omega \psi, \mathbb{1}_\Omega \varphi) = h_\xi(\psi, \mathbb{1}_\Omega \varphi). \end{aligned}$$

Now another application of (8) gives

$$(\xi, \mathbb{1}_\Omega Y_\psi \varphi) = h_{\mathbb{1}_\Omega \xi}(\psi, \varphi) = h_\xi(\psi, \mathbb{1}_\Omega \varphi) = (\xi, Y_\psi \mathbb{1}_\Omega \varphi)$$

for all $\xi, \varphi \in D(h) \cap L_\infty(\mathbf{R}^d)$. Hence

$$\mathbb{1}_\Omega Y_\psi \varphi = Y_\psi \mathbb{1}_\Omega \varphi$$

for all $\varphi \in D(h) \cap L_\infty(\mathbf{R}^d)$ and all $\psi \in C_c^\infty(\mathbf{R}^d)$. In particular this is valid for all $\varphi \in W^{1,2}(\mathbf{R}^d) \cap L_\infty(\mathbf{R}^d)$. But it follows from Lemma 3 that $T_t^\psi W^{1,2}(\mathbf{R}^d) \cap L_\infty(\mathbf{R}^d) = W^{1,2}(\mathbf{R}^d) \cap L_\infty(\mathbf{R}^d)$ for all $t \in \mathbf{R}$. Therefore the Duhamel formula gives

$$[T_t^\psi, \mathbb{1}_\Omega] \varphi = - \int_0^t ds T_{t-s}^\psi [Y_\psi, \mathbb{1}_\Omega] T_s^\psi \varphi = 0$$

for all φ in the L_2 -dense subspace $W^{1,2}(\mathbf{R}^d) \cap L_\infty(\mathbf{R}^d)$. Therefore Condition II now follows from Lemma 4.

II \Rightarrow I Let y_ψ denote the linear functional

$$\varphi \in C_c^\infty(\mathbf{R}^d) \mapsto y_\psi(\varphi) = \int_{\mathbf{R}^d} dx (Y_\psi \varphi)(x)$$

where $\psi \in C_c^\infty(\mathbf{R}^d) \subseteq D(H)$ is held fixed. It follows from the definition of Y_ψ that

$$y_\psi(\varphi) = h(\psi, \varphi) = (H\psi, \varphi)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. Then since $|(H\psi, \varphi)| \leq \|H\psi\|_\infty \|\varphi\|_1$ one deduces that y_ψ extends to a continuous linear functional over $L_1(\mathbf{R}^d)$ satisfying the bounds $|y_\psi(\varphi)| \leq \|H\psi\|_\infty \|\varphi\|_1$.

Condition II implies, however, that $T_t^\psi \mathbb{1}_\Omega = \mathbb{1}_\Omega$ on $L_\infty(\mathbf{R}^d)$ for all $t \in \mathbf{R}$ by Lemma 4. Therefore $T_t^\psi \mathbb{1}_\Omega \varphi = \mathbb{1}_\Omega T_t^\psi \varphi$ for all $t \in \mathbf{R}$ on $L_1(\mathbf{R}^d)$. Thus if $\varphi \in D(Y_\psi)$ then $\mathbb{1}_\Omega \varphi \in D(Y_\psi)$ and $Y_\psi \mathbb{1}_\Omega \varphi = \mathbb{1}_\Omega Y_\psi \varphi$ on $L_1(\mathbf{R}^d)$. Therefore

$$(9) \quad \begin{aligned} (H\psi, \mathbb{1}_\Omega \varphi) &= \int_{\mathbf{R}^d} dx (Y_\psi \mathbb{1}_\Omega \varphi)(x) \\ &= \int_{\mathbf{R}^d} dx (\mathbb{1}_\Omega Y_\psi \varphi)(x) = \int_{\Omega} dx (Y_\psi \varphi)(x) \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. In particular

$$|(H^{1/2}(H^{1/2}\psi), \mathbb{1}_\Omega \varphi)| = |(H\psi, \mathbb{1}_\Omega \varphi)| \leq \|Y_\psi \varphi\|_1 \leq h(\psi)^{1/2} h(\varphi)^{1/2}.$$

Then since $C_c^\infty(\mathbf{R}^d)$ is a core of H this bound extends to all $\varphi \in D(h)$. Hence $\mathbb{1}_\Omega D(h) \subseteq D(H^{1/2}) = D(h)$. Then Condition I follows from [FOT94], Theorem 1.6.1. \square

Theorem 2 is now a corollary of Theorem 1 and the following proposition.

Proposition 6. *Assume $c_{ij} \in W^{1,\infty}(\mathbf{R}^d)$. Let Ω be a measurable subset of \mathbf{R}^d .*

Then the following conditions are equivalent.

- I. $T_t^{(i)} L_2(\Omega) = L_2(\Omega)$ for all $i \in \{1, \dots, d\}$ and $t \in \mathbf{R}$.
- II. $T_t^\psi L_2(\Omega) = L_2(\Omega)$ for all $\psi \in C_c^\infty(\mathbf{R}^d)$ and $t \in \mathbf{R}$.

Proof. Note that we have omitted the index used in Section 2 to denote the space on which the flows act but this should not cause confusion as the following argument only involves $L_2(\mathbf{R}^d)$ and $L_\infty(\mathbf{R}^d)$.

First, Condition I is equivalent to the conditions $T_t^{(i)} \mathbb{1}_\Omega = \mathbb{1}_\Omega$ for all $i \in \{1, \dots, d\}$ and $t \in \mathbf{R}$ by Lemma 4. But the latter conditions are equivalent to

$$(10) \quad (Y_i^* \varphi, \mathbb{1}_\Omega) = 0$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$ and $i \in \{1, \dots, d\}$ by Corollary 5.

Secondly, Condition II is equivalent to

$$(11) \quad (Y_\psi^* \varphi, \mathbb{1}_\Omega) = 0$$

for all $\varphi, \psi \in C_c^\infty(\mathbf{R}^d)$ by similar reasoning.

Now if $\psi \in C_c^\infty(\mathbf{R}^d)$ then

$$(Y_\psi^* \varphi, \mathbb{1}_\Omega) = \sum_{j=1}^d (Y_j^* (\varphi \partial_j \psi), \mathbb{1}_\Omega)$$

for all $\varphi \in C_c^\infty(\mathbf{R}^d)$. Hence (10) implies (11). Conversely if for $\varphi \in C_c^\infty(\mathbf{R}^d)$ and $i \in \{1, \dots, d\}$ one chooses $\psi \in C_c^\infty(\mathbf{R}^d)$ such that $\psi(x) = x_i$ for $x \in \text{supp } \varphi$ then (11) implies (10). \square

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